

WITTEN DEFORMATION AND THE EQUIVARIANT INDEX

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ABSTRACT. Let M be a compact Riemannian manifold endowed with an isometric action of a compact, connected Lie group. The method of the Witten deformation is used to compute the virtual representation-valued equivariant index of a transversally elliptic, first order differential operator on M . The multiplicities of irreducible representations in the index are expressed in terms of local quantities associated to the isolated singular points of an equivariant bundle map that is locally Clifford multiplication by a Killing vector field near these points.

1. INTRODUCTION

The purpose of this paper is to compute the equivariant index multiplicities of an equivariant, transversally elliptic operator on a compact G -manifold, where G is a compact, connected Lie group. We use the method of Witten deformation to express the index in terms of combinatorial data associated to a given equivariant bundle map.

We start by establishing notation and reviewing the definitions of various types of equivariant indices associated to first order, transversally elliptic differential operators. In Section 1.2 we explain our application of the Witten deformation technique for calculating these equivariant indices and discuss the main results of this paper.

1.1. Types of equivariant indices. Suppose that a compact Lie group G acts by isometries on a compact, connected Riemannian manifold M , and let $E = E^+ \oplus E^-$ be a graded, G -equivariant, Hermitian vector bundle over M . We consider a first order G -equivariant differential operator $D^+ : \Gamma(M, E^+) \rightarrow \Gamma(M, E^-)$ which is elliptic merely in the directions transversal to the orbits of G , and let D^- be the formal adjoint of D^+ . Then the operator D^+ belongs to the class of *transversally elliptic differential operators* introduced by M. Atiyah in [1]. In this paper, we will assume for the most part that G is connected and that the operator D^+ is in addition transversally elliptic with respect to the action of a maximal torus in G .

The group G acts in a natural way on $\Gamma(M, E^\pm)$, and the (possibly infinite-dimensional) subspaces $\ker(D^+)$ and $\ker(D^-)$ are G -invariant subspaces. Thus, each of $\Gamma(M, E^\pm)$, $\ker(D^+)$, and $\ker(D^-)$ decomposes as a direct sum of irreducible representation spaces. Let $\rho : G \rightarrow \text{End}(V_\rho)$ be an irreducible unitary representation of G , and let $\chi_\rho : G \rightarrow \mathbb{C}$ be its character; that is, $\chi_\rho(g) = \text{tr}(\rho(g))$. By the Peter-Weyl Theorem, the functions $\{\chi_\rho\}_\rho$ are eigenfunctions of the Laplacian on G and form an orthonormal set in $L^2(G)$ with the normalized, biinvariant metric. Let $\Gamma(M, E^\pm)^\rho$ be the subspace of sections that is the direct sum

Date: October, 2006.

1991 Mathematics Subject Classification. 58J20; 58J37; 58J50.

Key words and phrases. equivariant index, group action, Witten deformation, perturbation, singularity, transversally elliptic, localization.

of the irreducible G -representation subspaces of $\Gamma(M, E^\pm)$ corresponding to representations that are unitarily equivalent to ρ . It can be shown that the operator

$$D^+ : \Gamma(M, E^+)^{\rho} \rightarrow \Gamma(M, E^-)^{\rho}$$

can be extended to a Fredholm operator between the appropriate Sobolev spaces, so that each irreducible representation of G appears with finite multiplicity in $\ker D^\pm$. Let $a_\rho^\pm \in \mathbb{Z}^+$ be the multiplicity of ρ in $\ker(D^\pm)$.

As in [1], we define the virtual representation-valued index of D to be

$$\text{ind}^G(D^+) := \sum_{\rho} (a_\rho^+ - a_\rho^-) [\rho],$$

where $[\rho]$ denotes the equivalence class of the irreducible representation ρ . The index multiplicity is

$$\text{ind}^\rho(D^+) := a_\rho^+ - a_\rho^- = \frac{1}{\dim V_\rho} \text{ind} \left(D^+|_{\Gamma(M, E^+)^{\rho} \rightarrow \Gamma(M, E^-)^{\rho}} \right).$$

In particular, if ρ_0 is the trivial representation of G , then

$$\text{ind}^{\rho_0}(D^+) = \text{ind} \left(D^+|_{\Gamma(M, E^+)^{G} \rightarrow \Gamma(M, E^-)^{G}} \right),$$

where the superscript G implies restriction to G -invariant sections.

The relationship between the index multiplicities and Atiyah's equivariant distribution-valued index $\text{ind}_g(D^+)$ is as follows. The virtual character $\text{ind}_g(D^+)$ is given by (see [1])

$$\begin{aligned} \text{ind}_g(D^+) &: = \text{“tr}(g|_{\ker D^+}) - \text{tr}(g|_{\ker D^-})\text{”} \\ &= \sum_{\rho} \text{ind}^\rho(D^+) \chi_\rho(g) \in \mathcal{D}(G), \end{aligned}$$

where $\mathcal{D}(G)$ is the set of distributions on G . Since $\ker D^+$ and $\ker D^-$ are in general infinite-dimensional, the sum above does not always converge, but it makes sense as a distribution on G . That is, if dg is the normalized, biinvariant volume form on G , and if $\phi = \sum c_\rho \chi_\rho \in C^\infty(G)$, then

$$\begin{aligned} \text{ind}_{(\bullet)}(D^+)(\phi) &= \int_G \phi(g) \overline{\text{ind}_g(D^+)} dg \\ &= \sum_{\rho} \text{ind}^\rho(D^+) \int \phi(g) \overline{\chi_\rho(g)} dg = \sum_{\rho} \text{ind}^\rho(D^+) c_\rho, \end{aligned}$$

an expression which converges because the coefficients c_ρ are rapidly decreasing and $\text{ind}^\rho(D^+)$ grows at most polynomially as ρ varies over the irreducible representations of G . From this calculation, we see that the multiplicities determine Atiyah's distributional index. Conversely, let $\alpha : G \rightarrow \text{End}(V_\alpha)$ be an irreducible unitary representation. Then

$$\text{ind}_{(\bullet)}(D^+)(\chi_\alpha) = \sum_{\rho} \text{ind}^\rho(D^+) \int \chi_\alpha(g) \overline{\chi_\rho(g)} dg = \text{ind}^\alpha D^+,$$

so that complete knowledge of the equivariant distributional index is equivalent to knowing all of the multiplicities $\text{ind}^\rho(D^+)$. Because the operator $D^+|_{\Gamma(M, E^+)^{\rho} \rightarrow \Gamma(M, E^-)^{\rho}}$ is Fredholm, all of the indices $\text{ind}^G(D^+)$, $\text{ind}_g(D^+)$, and $\text{ind}^\rho(D^+)$ depend only on the equivariant homotopy class of the principal transverse symbol of D^+ .

1.2. Content of the paper: applications of Witten deformation to equivariant index theory. About 25 years ago E. Witten [19] introduced a new way of proving Morse inequalities based on a deformation of the de Rham complex. His ideas were fruitfully applied in many specific situations. The purpose of this paper is to utilize this method to prove an explicit formula for the index $\text{ind}^\rho(D^+)$ in terms of data associated to the singular set of an equivariant bundle map $Z : E \rightarrow E$. In this paper, we require that the singularities, if they exist, are isolated and that the map Z has the form

$$Z = c(iV)$$

near each singular point, where c denotes a locally defined Clifford multiplication and V is a Killing vector field. Witten used a similar approach in [20] to prove the Atiyah-Hirzebruch vanishing theorem (see [4]) by showing $\text{ind}^\rho(D^+) = 0$ if D^+ is the Dirac operator on spinors and $G = S^1$; in his argument $Z = c(i\partial_\theta)$ globally. It should be mentioned that this idea is related to Atiyah's earlier method of "pushing a symbol," to extract information about the distribution-valued equivariant index near fixed points of a torus action (see [1, Chapter 6]).

Let $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$. We consider the following family of transversally elliptic operators, depending on a real parameter s :

$$\begin{aligned} D_s &= D + sZ, \text{ so that} \\ D_s^2 &= D^2 + s(DZ + ZD) + s^2Z^2 \end{aligned}$$

We want to study the spectral asymptotics of this family as $s \rightarrow \infty$. Unlike most other applications of Witten deformation where the operator $B = DZ + ZD$ is bounded (see [19], [15]), in this paper the operator B is first order (at least near singular points) and thus unbounded. In order to circumvent this difficulty, we require that the restriction of the B to $\Gamma(M, E)^\rho$ is a bundle map, which is indeed true if the first order part of B is a tangential derivative. In Section 2, we extend the localization theorem of Shubin ([18]) to the setting of transversally elliptic operators. This result allows us to reduce the computation of $\text{ind}^\rho(D^+)$ to investigating the spectrum of a certain model operator at each singular point of Z .

In Section 3, we restrict to the case where G is a torus, and we compute the spectral asymptotics of the operator $\frac{1}{s}D_s^2$ as $s \rightarrow \infty$ in terms of local information at each singular point of Z . The main result of the section is Theorem 3.8.

In Section 4, we apply Theorem 3.8 to evaluate the index $\text{ind}^\rho(D^+)$ in the case where G is a torus. The main result of the paper is the formula for this index in the Transverse Index Theorem, Theorem 4.2. In Section 5, we show that for any compact, connected Lie group G , the index $\text{ind}^\rho(D^+)$ can be expressed in terms of the corresponding indices for its maximal torus, as long as the relevant torus multiplicities are finite (as in the case where D^+ is also transversally elliptic with respect to the torus action).

Finally, in Section 6, we demonstrate applications of Theorem 4.2 to the signature and de Rham operators on G -manifolds and to a specific transversally elliptic operator on the sphere. These investigations yield an interesting new identity involving Killing vector fields on G -manifolds along with new proofs of other known identities; see Proposition 6.1. We also apply the theory in Section 5 to an example of an $SU(2)$ -action on a sphere.

1.3. Historical Comments. A large body of work over the last twenty years has yielded theorems that express $\text{ind}_g(D^+)$ and the corresponding local heat kernel supertrace in terms of topological and geometric quantities (as in the Atiyah-Segal-Singer index theorem for

elliptic operators or the Berline-Vergne Theorem for transversally elliptic operators — see [5],[7],[8]). The problem of expressing $\text{ind}^\rho(D^+)$ explicitly as a sum of topological or geometric quantities which are determined at the different strata of the G -manifold M is addressed in the paper [11]. The special cases where G is finite or when all of the isotropy groups have the same dimension were solved by M. Atiyah in [1], and it turns out both of these are special cases of the Orbifold Index Theorem by T. Kawasaki (see [13]). In the case when D^+ is elliptic, the Atiyah-Bott fixed point formula may be used to calculate the equivariant indices corresponding to a torus action from fixed point data, as in this paper (see [2],[3]). Much work has also been done on symplectic manifolds, where the local data comes from the critical set of the moment map. For example, see [17] for an analytic proof of the Guillemin-Sternberg conjecture ([12]). Also, see [14] and [9] for another Witten deformation approach to finding the equivariant index of a specific transversally elliptic symbol on a noncompact manifold.

2. EQUIVARIANT LOCALIZATION

Suppose a compact Lie group G acts by isometries on a closed, oriented Riemannian manifold M of dimension $2n$. Let E be a G -equivariant Hermitian bundle over M . Let ρ be an irreducible representation of G , and let $\Gamma(M, E)^\rho$ denote the space of sections of E of type ρ . For $s > 0$, let $H_s : \Gamma(M, E) \rightarrow \Gamma(M, E)$ be a transversally elliptic, G -equivariant, essentially self-adjoint, second order differential operator of the form

$$H_s = \frac{1}{s}A + B + sC,$$

where

- (1) A is a second order, transversally elliptic differential operator with positive definite principal transverse symbol.
- (2) For each irreducible representation ρ of G , $B|_{\Gamma(M, E)^\rho}$ is a bundle map.
- (3) C is a bundle map such that $C(x) \geq 0$ for all $x \in M$, and at each point \bar{x} where $C(\bar{x})$ is singular, there exists $c > 0$ such that

$$\begin{aligned} C(\bar{x}) &= 0, \bar{x}g = \bar{x} \text{ for all } g \in G, \text{ and} \\ C(x) &\geq c \cdot d(x, \bar{x})^2 \mathbf{1} \end{aligned}$$

in a neighborhood of \bar{x} , where $d(x, \bar{x})$ is the distance from x to \bar{x} .

- (4) A is elliptic in a neighborhood of each singular point of C .

Let H_s^ρ denote the restriction of H_s to $\Gamma(M, E)^\rho$. For each ρ , the operator H_s^ρ has discrete spectrum (see [1, p. 12-13]); this implies that the spectrum of H_s consists of a discrete set of eigenvalues, although some eigenvalues may have infinite multiplicities.

Near each singular point \bar{x} of C , we choose coordinates $x = (x_1, \dots, x_{2n})$ such that \bar{x} corresponds to the origin, $T_{\bar{x}}M = \mathbb{R}^{2n}$, and the volume form at the origin is $dx_1 \dots dx_{2n}$. We choose a trivialization of E near \bar{x} so that A , B , and C become differential operators with

matrix coefficients. We define the model operator $K_{\bar{x}}^{\rho} : \Gamma(\mathbb{R}^{2n}, E_{\bar{x}})^{\rho} \rightarrow \Gamma(\mathbb{R}^{2n}, E_{\bar{x}})^{\rho}$ by

$$\begin{aligned} K_{\bar{x}}^{\rho} &= \tilde{A} + \tilde{B}^{\rho} + \tilde{C}, \text{ where} \\ \tilde{A} &= \text{the principal part of } A \text{ at } \bar{x} \\ \tilde{B}^{\rho} &= B|_{\Gamma(M, E)^{\rho}}(\bar{x}) \\ \tilde{C} &= \sum x_i x_j (\nabla_i \nabla_j C)_{\bar{x}} = \text{the quadratic part of } C \text{ at } \bar{x}, \end{aligned}$$

where ∇ is the induced connection on $E \otimes E^*$. It is easy to check that \tilde{C} is independent of the coordinates chosen. Let dg denote differential of the action of $g \in G$ at \bar{x} , so we write $dg : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$. Let the action of g on $\mathbb{R}^{2n} \times E_{\bar{x}}$ be defined as

$$(x, v_{\bar{x}})g = (dg(x), g \cdot v_{\bar{x}}).$$

Lemma 2.1. *The operator $K_{\bar{x}}^{\rho}$ is equivariant with respect to this G -action.*

Proof. Since H_s is equivariant for each $s > 0$, it is elementary to show that each of the operators A , B , and C is equivariant. Then the principal symbol of A is equivariant, and in particular the principal symbol of A at \bar{x} is G -invariant. Thus, \tilde{A} is G -invariant. Next, since $B|_{\Gamma(M, E)^{\rho}}$ is equivariant, its restriction \tilde{B}^{ρ} to \bar{x} is also. Finally, since C is equivariant and the connection is equivariant, it follows that \tilde{C} is equivariant. \square

Lemma 2.2. *For each irreducible representation ρ of G and each fixed point \bar{x} of G , the operator $K_{\bar{x}}^{\rho} : \Gamma(\mathbb{R}^{2n}, E_{\bar{x}})^{\rho} \rightarrow \Gamma(\mathbb{R}^{2n}, E_{\bar{x}})^{\rho}$ has discrete spectrum.*

Proof. Consider the extended operator $K_{\bar{x}}^{\rho} : \Gamma(\mathbb{R}^{2n}, E_{\bar{x}}) \rightarrow \Gamma(\mathbb{R}^{2n}, E_{\bar{x}})$. This operator is elliptic and essentially self-adjoint, and the operator is bounded below by $(C_1 + C_2 \cdot |x|^2) \mathbf{1}$, where $C_1 \in \mathbb{R}$ and $C_2 > 0$. Since this bound goes to infinity as $x \rightarrow \pm\infty$, the operator $K_{\bar{x}}^{\rho} - (C_1 - 1) \mathbf{1}$ has a compact resolvent. Thus, the restriction of $K_{\bar{x}}^{\rho}$ to $\Gamma(\mathbb{R}^{2n}, E_{\bar{x}})^{\rho}$ also has a compact resolvent. \square

We define the model operator K^{ρ} by

$$K^{\rho} = \bigoplus_{\text{fixed point } \bar{x}} K_{\bar{x}}^{\rho}.$$

Clearly, this operator has discrete spectrum. Let

$$\mu_1^{\rho} < \mu_2^{\rho} < \mu_3^{\rho} < \dots$$

be the distinct eigenvalues of K^{ρ} with corresponding multiplicities $m_1^{\rho}, m_2^{\rho}, m_3^{\rho}, \dots$

Theorem 2.3. *(Equivariant Localization Theorem) For each irreducible representation ρ of G and for each fixed $N > 0$, there exists $c > 0$ and $s_0 > 0$ such that for any $s > s_0$ and any $j \leq N$, the interval $(\mu_j^{\rho} - cs^{-1/5}, \mu_j^{\rho} + cs^{-1/5})$ contains exactly m_j^{ρ} eigenvalues of H_s^{ρ} . Furthermore, all the eigenvalues of H_s^{ρ} contained in $(-\infty, \mu_N^{\rho} + cs^{-1/5})$ are contained in*

$$\bigcup_{j \leq N} (\mu_j^{\rho} - cs^{-1/5}, \mu_j^{\rho} + cs^{-1/5}).$$

Proof. We show how to generalize Theorem 1.1 in [18] to the equivariant setting. We identify the parameter s in our theorem with $\frac{1}{h}$ in [18].

To obtain an upper bound for the eigenvalues of H_s^{ρ} (or a lower bound on the spectral counting function of H_s^{ρ}), we use eigensections of the model operator K^{ρ} to produce test

sections for H_s^ρ . Suppose that ψ is an eigensection of $K_{\bar{x}}^\rho : \Gamma(\mathbb{R}^{2n}, E_{\bar{x}})^\rho \rightarrow \Gamma(\mathbb{R}^{2n}, E_{\bar{x}})^\rho$ corresponding to the eigenvalue λ . Let $J \in C_0^\infty(\mathbb{R}^{2n})$ be a radial function defined such that $0 \leq J \leq 1$, $J(x) = 1$ if $|x| \leq 1$, $J(x) = 0$ if $|x| \geq 2$. For any $s > 0$, let $J^{(s)}(x) = J(s^{2/5}x)$. Then the section

$$\phi(x) = J^{(s)}(x) s^{n/2} \psi(s^{1/2}x)$$

is in $\Gamma(\mathbb{R}^{2n}, E_{\bar{x}})^\rho$ as well, because $J^{(s)}$ is G -invariant. We produce a corresponding element $\tilde{\phi} \in \Gamma(M, E)^\rho$ that has support in a small neighborhood U of \bar{x} as follows. Let γ be the unit speed geodesic from \bar{x} to $p \in U$, let x_p be the geodesic normal coordinates of p , and let $P_\gamma : E_{\bar{x}} \rightarrow E_p$ denote parallel translation along γ . We define

$$\tilde{\phi}(p) = P_\gamma \phi(x_p).$$

Clearly, $\tilde{\phi} \in \Gamma(M, E)$. Because the connection on E is equivariant, parallel translation commutes with the action of G , and $\tilde{\phi} \in \Gamma(M, E)^\rho$. This specific trivialization of E produces test sections that can be used as in [18] to obtain the upper bounds for the eigenvalues of H_s^ρ . We denote $\Phi : \Gamma(\mathbb{R}^{2n}, E_{\bar{x}})^\rho \rightarrow \Gamma(U, E)^\rho$ to be the trivialization $\phi \rightarrow \tilde{\phi}$.

To obtain a lower bound on the eigenvalues of H_s^ρ (or an upper bound on the spectral counting function of H_s^ρ), we proceed exactly as in [18]. The functions in the partition of unity are chosen so that those corresponding to neighborhoods of singular points are radial; then the partition of unity will consist of invariant functions. Next, the IMS localization formula allows us to localize to these small neighborhoods, comparing the operators $\Phi^{-1} H_s^\rho \Phi$ and K^ρ . \square

3. ANALYSIS OF EQUIVARIANT PERTURBATIONS

In this section, we are going to apply Theorem 2.3 to the following situation. Let $G = T^m \cong \mathbb{R}^m / 2\pi\mathbb{Z}^m$ act on the right by isometries on a closed, oriented Riemannian manifold M of dimension $2n$. Let $D^+ : \Gamma(M, E^+) \rightarrow \Gamma(M, E^-)$ be a first-order, G -equivariant, transversally elliptic operator, where E^+ and E^- are G -equivariant Hermitian vector bundles of rank $2r$ over M . Let $E = E^+ \oplus E^-$, and let $D : \Gamma(M, E) \rightarrow \Gamma(M, E)$ denote the operator $(D^+, (D^+)^*)$, where $*$ denotes the adjoint.

Consider the following family of operators, depending on a real parameter s :

$$D_s = D + sZ,$$

where Z has the following properties:

- (1) $Z : E^\pm \rightarrow E^\mp$ is a smooth, self-adjoint, equivariant bundle map that is nonsingular away from a finite number of points of M .
- (2) For each irreducible representation ρ , the restriction of $DZ + ZD$ to $\Gamma(M, E)^\rho$ is a bundle map.
- (3) In a small neighborhood $U_{\bar{x}}$ of each singular point \bar{x} of Z , we assume that E^\pm has the structure of an equivariant Clifford bundle (with equivariant Clifford connection ∇) and that D is a(n equivariant) Dirac operator near these points (see [6]).
- (4) In $U_{\bar{x}}$, the operator $D + sZ$ has the following explicit form. We require

$$Z = c(iV), \text{ so that } D_s = D + sc(iV),$$

where V is a vector field induced from some element $\mathbf{v}_{\bar{x}}$ of the Lie algebra \mathfrak{g} of the torus G such that the closure of $\{\exp(t\mathbf{v}_{\bar{x}}) \mid t \in \mathbb{R}\}$ is the entire torus T^m , and where $c(iV)$ denotes Clifford multiplication by iV .

For example, if D is a Dirac operator on sections of a Clifford bundle and V is a global Killing vector field with isolated fixed points that induces an infinitesimal isometry of the bundle, then the operator $Z = c(iV)$ satisfies the conditions above (see Lemma 3.1), where the torus group is the closure of the flow of V in the isometry group of M . For a case of a transversally elliptic operator and perturbation Z , see Example 6.3.

The proof of the next lemma can be found in the Appendix.

Lemma 3.1. *In the notation above, for any vector field V , if D is a Dirac operator,*

$$(D_s)^2 = D^2 + s(-2i\nabla_V - i\operatorname{div}(V) + ic(d(V^*))) + s^2|V|^2.$$

Here V^ is the one-form dual to the vector field V , and $c(\alpha \wedge \beta) := c(\alpha)c(\beta)$ for orthogonal covectors α and β .*

In what follows, we need to define the Lie derivative of a section of E . Since G acts on M on the right and since E is G -equivariant, the bundle E is endowed with the lifted left action $F_g : E_x \rightarrow E_{xg}$ on E for each $g \in G$.

Definition 3.2. *The induced action ψ_g of $g \in G$ on the a section $u \in \Gamma(M, E)$ is*

$$(\psi_g u)(x) = F_{g^{-1}}(u(xg)),$$

and the action satisfies

$$\psi_{gh} = \psi_h \circ \psi_g$$

for all $g, h \in G$.

Definition 3.3. *The Lie derivative $\mathcal{L}_V u$ of a section $u \in \Gamma(M, E)$ in direction V (as above, the vector field induced from $\mathbf{v} \in \mathfrak{g}$) is*

$$(\mathcal{L}_V u)(x) = \left. \frac{d}{dt} [F_{\exp(-t\mathbf{v})}(u(x \exp(t\mathbf{v}))) \right]_{t=0}.$$

With this definition, \mathcal{L}_V satisfies the usual properties of Lie derivative on tensors. For example, the standard induced action of a Lie group on vector fields and forms gives the ordinary Lie derivative. The following lemma is standard.

Lemma 3.4. *If V is an infinitesimal isometry, then the operator $A_V = \nabla_V - \mathcal{L}_V$ is a skew-Hermitian endomorphism of E .*

Example 3.5. *Let V be a Killing field generating an action by isometries on a Riemannian manifold (M, g) . If M is in addition a spin manifold, then the action automatically lifts to the spinor bundle S . If we let \mathcal{L}_V^S be the Lie derivative of this action on the spinors, induced by the action on the frame bundle, then*

$$A_V = \nabla_V^S - \mathcal{L}_V^S = \frac{1}{4}c(d(V^*)).$$

See the proof in the appendix.

Corollary 3.6. *If D is a Dirac operator, and if $D_s = D + sc(iV)$ with V an infinitesimal isometry as above, then*

$$(D_s)^2 = D^2 + s(-2i\mathcal{L}_V - 2iA_V + ic(d(V^*))) + s^2|V|^2. \quad (3.1)$$

Proof. Combine the two previous lemmas, and observe that for a Killing vector field V we have $\operatorname{div}(V) = 0$. \square

Remark 3.7. *A similar computation was done by Bismut and explained in [6, Chapter 8] in the heat kernel proof of the Kirillov character formula. In this computation, the endomorphism $-A_X$ is called the “moment” of the connection, used in the context of frame bundles and bundles of forms.*

The nondegeneracy of V at the zero \bar{x} implies $|V(x)|^2 \geq c \cdot d(x, \bar{x})^2$ for x near \bar{x} , so the lemmas above imply that the hypotheses of Theorem 2.3 are satisfied for the operator

$$H_s = \frac{1}{s} (D_s)^2. \quad (3.2)$$

Fix $\rho : T^m \rightarrow \mathbb{C}$ to be a particular irreducible unitary representation. Note that if we choose coordinates $\theta = (\theta_1, \dots, \theta_m) \in (\mathbb{R}/2\pi\mathbb{Z})^m$, then the representation has the form

$$\rho(\theta) = e^{i\mathbf{b} \cdot \theta}, \quad (3.3)$$

where $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{Z}^m$. Note that the vector \mathbf{b} depends on the choice of coordinates θ ; for instance, if θ_i is replaced by $-\theta_i$, then b_i is replaced by $-b_i$. In what follows, the choice of coordinates θ will depend on $\mathbf{v}_{\bar{x}}$.

Fix a singular point \bar{x} of Z . We now describe the model operator $K_{\bar{x}}^\rho$ and compute its eigenvalues.

We will use geodesic normal coordinates centered at \bar{x} . In these coordinates, \tilde{A} , the principal part of $A = D^2$ at \bar{x} , is the Euclidean Laplacian.

Now we compute

$$\tilde{B}^\rho = B|_{\Gamma(M,E)^\rho}(\bar{x}) = (-2i\mathcal{L}_V - 2iA_V + ic(d(V^*)))|_{\Gamma(M,E)^\rho}(\bar{x})$$

The vector $\mathbf{v}_{\bar{x}} \in \mathfrak{g}$ generates a dense flow $\theta(t)$ on T^m by the formula

$$\theta(t) = \exp(t\mathbf{v}_{\bar{x}}) = t\tau = (t\tau_1, t\tau_2, \dots, t\tau_m) \in T^m,$$

where $\tau = (\tau_1, \dots, \tau_m) \in \mathbb{R}^m$. We choose the coordinates θ so that the torus action satisfies $\tau_p > 0$ for $1 \leq p \leq m$. Since the flow is dense, the set $\{\tau_1, \dots, \tau_m\}$ is linearly independent over \mathbb{Q} .

The representation ρ and choice of coordinates θ uniquely determine the vector $\mathbf{b} \in \mathbb{Z}^m$ as in formula (3.3). If $u \in \Gamma(M, E)$ is of type ρ , then near \bar{x} we have

$$\mathcal{L}_V u = i(\mathbf{b} \cdot \tau) u.$$

The action of $\theta \in T^m$ on a small neighborhood of the point \bar{x} can be transferred to the tangent space $T_{\bar{x}}M$ via conjugation with the exponential map; the induced action on $T_{\bar{x}}M$ is an isometry.

Choose orthonormal coordinates $(x_1, y_1, \dots, x_n, y_n) = (z_1, \dots, z_n)$ on $T_{\bar{x}}M \cong \mathbb{C}^n$ so that $\theta \in T^m$ acts by

$$(z_1, \dots, z_n) \theta = (e^{i\mathbf{k}_1 \cdot \theta} z_1, \dots, e^{i\mathbf{k}_n \cdot \theta} z_n), \quad (3.4)$$

where each $\mathbf{k}_l = (k_{l1}, \dots, k_{lm}) \in \mathbb{Z}^m$. We assume in addition that for each l ,

$$\kappa_l := \mathbf{k}_l \cdot \tau > 0; \quad (3.5)$$

otherwise, replace x_l with y_l and vice versa. Note that the resulting coordinates will not necessarily have the same orientation as the induced orientation that comes from the manifold M .

Next, we choose an Hermitian coordinates (w_1, \dots, w_r) of $E_{\bar{x}}$ so that the action of $\theta \in T^m$ on $E_{\bar{x}}$ is given by

$$F_{\theta}(w_1, \dots, w_r) = (e^{i\mathbf{a}_1 \cdot \theta} w_1, \dots, e^{i\mathbf{a}_r \cdot \theta} w_r), \quad (3.6)$$

with $\mathbf{a}_j = (a_{j1}, \dots, a_{jm}) \in \mathbb{Z}^m$. Further, we choose the basis of $E_{\bar{x}} = E_{\bar{x}}^+ \oplus E_{\bar{x}}^-$ so that the grading operator is diagonal in this basis. (Note that the grading commutes with the group action, so we may do this.)

We compute that

$$\begin{aligned} V &= \sum_{l=1}^n \kappa_l \partial_{\phi_l}, \quad |V|^2 = \sum_{l=1}^n \kappa_l^2 |z_l|^2 \\ V^* &= \sum_{l=1}^n \kappa_l |z_l|^2 d\phi_l, \quad dV^* = 2 \sum_{l=1}^n \kappa_l d\text{vol}_l, \\ A_V &= \nabla_V - \mathcal{L}_V = i \sum_{j=1}^r (\mathbf{a}_j \cdot \tau) P_j, \end{aligned}$$

where ∂_{ϕ_l} is the angular vector field $x_l \partial_{y_l} - y_l \partial_{x_l}$, $d\text{vol}_l = dx_l \wedge dy_l$, and $P_j =$ projection onto the w_j plane (i.e. j th coordinate plane) in $E_{\bar{x}}$.

Observe that the operators $ic(d\text{vol}_l)$ mutually commute, commute with the chirality operator and with the group action, and square to 1. Since the operators $[ic(d\text{vol}_l)]$ commute with the group action, they commute with each P_j . Let $\varepsilon_{jl} \in \{-1, 1\}$ be defined by

$$\varepsilon_{jl} P_j = ic(d\text{vol}_l) P_j \quad (3.7)$$

Using these calculations, we obtain the second term of the model operator $K_{\bar{x}}^{\rho}$:

$$\begin{aligned} \tilde{B}^{\rho} &= (-2i\mathcal{L}_V - 2iA_V + ic(d(V^*)))|_{\Gamma(M,E)^{\rho}}(\bar{x}) \\ &= 2\mathbf{b} \cdot \tau + 2 \sum_{j=1}^r (\mathbf{a}_j \cdot \tau) P_j + 2 \sum_{l=1}^n \kappa_l [ic(d\text{vol}_l)] \end{aligned}$$

Finally, we must compute

$$\tilde{C} = \text{the quadratic part of } |V|^2 \text{ at } \bar{x} = \sum_{l=1}^n \kappa_l^2 |z_l|^2.$$

Thus, the model operator relevant to Theorem 2.3 is

$$\begin{aligned} K_{\bar{x}}^{\rho} &= \sum_{l=1}^n (-\partial_{x_l}^2 - \partial_{y_l}^2) + \left(2\mathbf{b} \cdot \tau + 2 \sum_{j=1}^r (\mathbf{a}_j \cdot \tau) P_j + 2 \sum_{l=1}^n \kappa_l [ic(d\text{vol}_l)] \right) + \sum_{l=1}^n \kappa_l^2 |z_l|^2 \\ &= \sum_{j=1}^r \left[\sum_{l=1}^n \left(-\partial_{x_l}^2 - \partial_{y_l}^2 + \sum_{l=1}^n \kappa_l^2 (x_l^2 + y_l^2) \right) \right. \\ &\quad \left. + \left(2\mathbf{b} \cdot \tau + 2 \sum_{j=1}^r (\mathbf{a}_j \cdot \tau) + 2 \sum_{l=1}^n \kappa_l \varepsilon_{jl} \right) \right] P_j \end{aligned} \quad (3.8)$$

It is well-known that for the sum of oscillators

$$\sum_{l=1}^n \left(-\partial_{x_l}^2 - \partial_{y_l}^2 + \sum_{l=1}^n \kappa_l^2 (x_l^2 + y_l^2) \right),$$

the eigenvalues are the numbers (determined by an arbitrary $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ and $\mathbf{d} = (d_1, \dots, d_n) \in (\mathbb{Z}_{\geq 0})^n$)

$$\lambda_{\mathbf{m}, \mathbf{d}} = 2 \sum_{l=1}^n \kappa_l (|m_l| + 2d_l + 1) ,$$

corresponding to the scalar eigenfunctions

$$\phi_{\mathbf{m}, \mathbf{d}} = \prod_{l=1}^n e^{-\frac{1}{2} r_l^2 \kappa_l} r_l^{|m_l|} e^{i m_l \theta_l} \cdot L_{d_l, |m_l|} (r_l^2 \kappa_l) ,$$

where (r_l, θ_l) are polar coordinates in the (x_l, y_l) -plane and $L_{d_l, |m_l|}(r)$ is a generalized Laguerre polynomial of degree $d_l \geq 0$. In particular, $L_{0, |m_l|}(r) = 1$ for all $m_l \in \mathbb{Z}$. It is well known that the set $\{\phi_{\mathbf{m}, \mathbf{d}} \mid \mathbf{m} \in \mathbb{Z}^n, \mathbf{d} \in (\mathbb{Z}_{\geq 0})^n\}$ is an orthogonal basis of $L^2(\mathbb{C}^n)$.

Next, we compute the action of $\theta \in T^m$ on each $\phi_{\mathbf{m}, \mathbf{d}}$:

$$\begin{aligned} \phi_{\mathbf{m}, \mathbf{d}}(z_1, \dots, z_n) &\mapsto \phi_{\mathbf{m}, \mathbf{d}}(e^{i \mathbf{k}_1 \cdot \theta} z_1, \dots, e^{i \mathbf{k}_n \cdot \theta} z_n) \\ &= \exp \left(i \sum_{j=1}^n m_j \mathbf{k}_j \cdot \theta \right) \phi_{\mathbf{m}, \mathbf{d}}(z_1, \dots, z_n) . \end{aligned}$$

For each $j \in \{1, \dots, r\}$, let e_j be a basis vector that spans $P_j(E_{\overline{x}})$. Since we wish to consider sections of type ρ , first observe that since $\theta \in T^m$ acts on a section $u \in \Gamma(\mathbb{R}^{2n}, \mathbb{C})$ by $\psi_{\theta}(u)(z) = F_{-\theta} u(z\theta)$,

$$\begin{aligned} \psi_{\theta}(\phi_{\mathbf{m}, \mathbf{d}} e_j)(z) &= F_{-\theta}(\phi_{\mathbf{m}, \mathbf{d}}((z)\theta) e_j) = F_{-\theta} \left(\exp \left(i \sum_{h=1}^n m_h \mathbf{k}_h \cdot \theta \right) \phi_{\mathbf{m}, \mathbf{d}}(z) e_j \right) \\ &= \exp \left(i \left[-\mathbf{a}_j + \sum_{h=1}^n m_h \mathbf{k}_h \right] \cdot \theta \right) \phi_{\mathbf{m}, \mathbf{d}}(z) e_j \end{aligned}$$

In order that $\phi_{\mathbf{m}, \mathbf{d}} P_j \in \Gamma(\mathbb{R}^{2n}, \mathbb{C})^{\rho}$, the following equation must be satisfied:

$$-\mathbf{a}_j + \sum_{h=1}^n m_h \mathbf{k}_h = \mathbf{b} .$$

We note that there are many choices of the integers m_h , in general an infinite number, that satisfy the equations above for given $\mathbf{a}_j, \mathbf{k}_h$, and \mathbf{b} . Taking the dot product with τ , we have

$$\begin{aligned} -\mathbf{a}_j \cdot \tau + \sum_{h=1}^n m_h (\mathbf{k}_h \cdot \tau) &= \mathbf{b} \cdot \tau, \text{ or} \\ \sum_{h=1}^n m_h \kappa_h &= \mathbf{a}_j \cdot \tau + \mathbf{b} \cdot \tau . \end{aligned} \tag{3.9}$$

The possible $\mathbf{m} \in \mathbb{Z}^n$ satisfying (3.9) are integer points in an $(n-1)$ -dimensional plane, since the right hand side is fixed. From (3.8), the restriction of $K_{\overline{x}}^{\rho}$ to a section $\phi_{\mathbf{m}, \mathbf{d}} e_j$ of type ρ

with a specific choice of the \mathbf{m} gives the formula

$$\begin{aligned} K_{\overline{x}}^\rho \phi_{\mathbf{m}, \mathbf{d}} e_j &= \left(2 \sum_{l=1}^n \kappa_l (|m_l| + 2d_l + 1) + 2\mathbf{b} \cdot \tau + 2\mathbf{a}_j \cdot \tau + 2 \sum_{l=1}^n \kappa_l \varepsilon_{jl} \right) \phi_{\mathbf{m}} e_j \\ &= \left(2 \sum_{l=1}^n \kappa_l (|m_l| + m_l + 2d_l + 1 + \varepsilon_{jl}) \right) \phi_{\mathbf{m}} e_j. \end{aligned}$$

We have proved the following theorem:

Theorem 3.8. *The spectrum of $K_{\overline{x}}^\rho$ is the set of real numbers of the form*

$$\lambda = 2 \sum_{l=1}^n \kappa_l (|m_l| + m_l + 2d_l + 1 + \varepsilon_{jl}),$$

where the multiplicity of the eigenvalue λ is the number of pairs $(\mathbf{m}, \mathbf{d}) \in \mathbb{Z}^n \times (\mathbb{Z}_{\geq 0})^n$ such that there exists $j \in \{1, \dots, r\}$ such that

$$\sum_{h=1}^n m_h \mathbf{k}_h = \mathbf{a}_j + \mathbf{b} \text{ and } 2 \sum_{l=1}^n \kappa_l (|m_l| + m_l + 2d_l + 1 + \varepsilon_{jl}) = \lambda.$$

Remark 3.9. *Note that the multiplicities of the eigenvalues above are finite, since*

$$\lambda = 2 \sum_{l=1}^n \kappa_l (|m_l| + 2d_l + 1 + \varepsilon_{jl}) + \mathbf{b} \cdot \tau + \mathbf{a}_j \cdot \tau,$$

and the quantities $|m_l|$ and d_l must be bounded.

Since

$$K^\rho = \bigoplus_{\text{singular point } \overline{x}} K_{\overline{x}}^\rho,$$

the spectrum $\sigma(K^\rho)$ satisfies

$$\sigma(K^\rho) = \bigcup_{\text{singular point } \overline{x}} \sigma(K_{\overline{x}}^\rho).$$

Remark 3.10. *Theorem 2.3, Theorem 3.8, and Equation (3.2) imply that as $s \rightarrow \infty$, the eigenvalues of $\frac{1}{s}(D + sZ)^2$ restricted to sections of type ρ approach the eigenvalues λ of the K^ρ as described above.*

4. APPLICATIONS TO EQUIVARIANT INDEX THEORY

We now apply Theorem 2.3 and Theorem 3.8 to compute the index $\text{ind}_{T^m}^\rho(D)$ of D restricted to sections of type ρ . Since the equivariant index does not depend on continuous perturbations, the index of D restricted to sections of type ρ is

$$\begin{aligned} \text{ind}_{T^m}^\rho(D) &= \text{ind}_{T^m}^\rho(D_s) \\ &= \dim \ker \left((D_s)^2|_{\Gamma(M, E^+)^{\rho}} \right) - \dim \ker \left((D_s)^2|_{\Gamma(M, E^-)^{\rho}} \right). \end{aligned}$$

We now calculate these kernels independently using Theorem 2.3 and Theorem 3.8. The standard argument implies the following lemma.

Lemma 4.1. *The index satisfies*

$$\text{ind}_{T^m}^\rho(D) = \sum_{\bar{x}} \dim \ker K_{\bar{x}}^{\rho,+} - \dim \ker K_{\bar{x}}^{\rho,-},$$

where the superscript \pm refers to the restriction to $E_{\bar{x}}^\pm$.

Next, $\dim \ker (K_{\bar{x}}^\rho)$ is the number of pairs $(\mathbf{m}, \mathbf{d}) \in \mathbb{Z}^n \times (\mathbb{Z}_{\geq 0})^n$ such that there exists $j \in \{1, \dots, r\}$ such that

$$\sum_{h=1}^n m_h \mathbf{k}_h = \mathbf{a}_j + \mathbf{b} \text{ and } 2 \sum_{l=1}^n \kappa_l (|m_l| + m_l + 2d_l + 1 + \varepsilon_{jl}) = 0.$$

In this formula, the quantities \mathbf{a}_j , \mathbf{k}_h , \mathbf{b} , κ_h all depend on the critical point \bar{x} . Since each κ_h is positive, $2 \sum_{h=1}^n \kappa_h (|m_h| + m_h + 2d_l + 1 + \varepsilon_{jh}) = 0$ if and only if each m_h is nonpositive, each d_l is zero, and each ε_{jh} is -1 for $1 \leq h \leq n$. Thus, we may express $\dim \ker (K_{\bar{x}}^\rho)$ as

$$\# \left\{ \mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n \mid m_h \leq 0 \text{ and there exists } j \in \{1, \dots, r\} \text{ such that } \varepsilon_{jh} = -1 \text{ for all } h, 1 \leq h \leq n \text{ and } \sum_{h=1}^n m_h \mathbf{k}_h = \mathbf{a}_j + \mathbf{b} \right\}$$

The theorem below follows immediately. Recall that the j^{th} coordinate plane as in Formula (3.7) is a subspace of $E_{\bar{x}}^+$ or of $E_{\bar{x}}^-$.

Theorem 4.2. *(Transverse Index Theorem) Let*

$$k_j(\bar{x}) = \begin{cases} \# \{ \mathbf{m} \in \mathbb{Z}^n \mid m_h \leq 0 \text{ and } \sum_{h=1}^n m_h \mathbf{k}_h = \mathbf{a}_j + \mathbf{b} \} & \text{if } \varepsilon_{jh} = -1 \text{ for all } h \\ 0 & \text{otherwise} \end{cases}$$

and let

$$\text{sign}(j) = \pm 1,$$

according to whether the j^{th} coordinate plane is in $E_{\bar{x}}^\pm$. Then

$$\text{ind}_{T^m}^\rho(D) = \sum_{Z(\bar{x})=0} \sum_{j=1}^r \text{sign}(j) k_j(\bar{x}).$$

5. INDEX MULTIPLICITIES FOR THE LIE GROUP AND ITS MAXIMAL TORUS.

Suppose that T^m is a maximal torus in a compact, connected Lie group G , and let D be a G -equivariant, transversally elliptic, first order differential operator that is also transversally elliptic with respect to the T^m action. Then there is a relationship between the multiplicities $\text{ind}_{T^m}^\rho(D)$ and $\text{ind}_G^\mu(D)$, for a given irreducible representation μ of G . We choose the coordinates $\theta \in \mathbb{R}^m / 2\pi\mathbb{Z}^m$ for the torus T^m . For any character ξ_α of a (not necessarily irreducible) representation α of G , the restriction of ξ_α of G to T^m yields a character of T^m . Let χ_μ denote the character of a specific irreducible unitary representation μ , which

has L^2 norm 1 with respect to the Haar measure. Since characters are class functions, the multiplicity n_μ^α of μ in α is

$$n_\mu^\alpha = \int_G \xi_\alpha(g) \overline{\chi_\mu(g)} dg = \int_{T^m} \widetilde{\xi}_\alpha(\theta) \overline{\widetilde{\chi}_\mu(\theta)} f(\theta) \frac{d\theta}{(2\pi)^m},$$

where $f(\theta)$ is the factor of the integrand from the Weyl Integration Formula and dg is the normalized, biinvariant volume form on G , and where the tilde $\widetilde{(\cdot)}$ denotes restriction to the torus. Since the characters $e^{i\mathbf{b}\cdot\theta}$ of the torus form an orthonormal basis of $L^2(T^m)$, we may write

$$\overline{\widetilde{\chi}_\mu(\theta)} f(\theta) = \sum_{\mathbf{b}} \beta_\mu^{\mathbf{b}} e^{-i\mathbf{b}\cdot\theta},$$

where $\beta_\mu^{\mathbf{b}}$ are complex numbers depending only on the irreducible representation μ . Then

$$\begin{aligned} \int_G \xi_\alpha(g) \overline{\chi_\mu(g)} dg &= \sum_{\mathbf{b}} \beta_\mu^{\mathbf{b}} \widetilde{n}_\mu^\alpha, \text{ where} \\ \widetilde{\xi}_\alpha(\theta) &= \sum_{\mathbf{b}} \widetilde{n}_\mu^\alpha e^{i\mathbf{b}\cdot\theta}, \end{aligned}$$

so that \widetilde{n}_μ^α is the multiplicity of $e^{i\mathbf{b}\cdot\theta}$ in the restriction of α to T^m . Thus, the multiplicities of G -irreducible representations in α are determined by the multiplicities of the T^m -irreducible representations of the restriction of α to a maximal torus. Thus, we have

$$\text{ind}_G^\mu(D) = \sum_{\mathbf{b}} \beta_\mu^{\mathbf{b}} \text{ind}_{T^m}^{\rho_{\mathbf{b}}} (D),$$

where $\rho_{\mathbf{b}}(\theta)$ is multiplication by $e^{i\mathbf{b}\cdot\theta}$. and so the index multiplicities for the Lie group G are determined in a universal way from the multiplicities of the maximal torus. Note that the formula above is valid even if D is not transversally elliptic with respect to T , as long as the multiplicities of the representations of type $\rho_{\mathbf{b}}$ in $\ker D$ and $\ker D^*$ with $\beta_\mu^{\mathbf{b}} \neq 0$ are finite.

We comment that this procedure is a consequence of the following. If a vector space is a unitary representation space of a compact, connected Lie group G , it is also a representation space of the maximal torus T . This vector space may be decomposed into irreducible representation spaces of G or into irreducible representation spaces of T . If the multiplicities of all of these irreducible representations are finite, then the G -multiplicities determine the T -multiplicities, and, surprisingly, the T -multiplicities determine the G -multiplicities.

Example 5.1. Suppose that $G = SU(2)$. We compute the coefficients $\beta_\mu^{\mathbf{b}}$ for a given irreducible unitary representation of $SU(2)$. We follow [10, pp. 84ff]. Let V_n be the space of homogeneous polynomials of degree n in $z = (z_1, z_2) \in \mathbb{C}^2$, and let

$$\mu_n : SU(2) \rightarrow \text{End}(V_n)$$

be defined for $g \in SU(2)$ by $(\mu_n(g)P)(z) = P(zg)$. These are precisely the irreducible unitary representations of $SU(2)$. Let $T < SU(2)$ be the maximal torus defined as

$$T = \left\{ E(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

The character χ_n of μ_n satisfies

$$\begin{aligned}\chi_n(E(t)) &= \widetilde{\chi}_\mu(E(t)) = \sum_{k=0}^n e^{i(n-2k)t} \\ &= \frac{\sin((n+1)t)}{\sin(t)} \text{ for } t \notin \pi\mathbb{Z}.\end{aligned}$$

For any class function ω on $SU(2)$ we have the Weyl integration formula

$$\int_{SU(2)} \omega(g) dg = \int_0^{2\pi} \omega(E(t)) (2 \sin^2(t)) \frac{dt}{2\pi},$$

so that the function f from this section is defined by

$$f(t) = 2 \sin^2(t)$$

For generic t ,

$$\begin{aligned}\overline{\widetilde{\chi}_n(E(t))} f(t) &= \sum_{b \in \mathbb{Z}} \beta_n^b e^{-ib\theta}, \text{ or} \\ \frac{\sin((n+1)t)}{\sin(t)} (2 \sin^2(t)) &= \sum_{b \in \mathbb{Z}} \beta_n^b e^{-ib\theta}.\end{aligned}$$

The left hand side is

$$2 \left(\frac{e^{i(n+1)t} - e^{-i(n+1)t}}{2i} \right) \left(\frac{e^{it} - e^{-it}}{2i} \right) = \frac{1}{2} e^{-int} - \frac{1}{2} e^{i(n+2)t} - \frac{1}{2} e^{-i(n+2)t} + \frac{1}{2} e^{int},$$

so that

$$\beta_n^b = \begin{cases} \frac{1}{2} & \text{if } b = n \text{ or } -n \\ -\frac{1}{2} & \text{if } b = n+2 \text{ or } -n-2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if D is an $SU(2)$ -equivariant, transversally elliptic, first order differential operator on a closed manifold such that D is also transversally elliptic with respect to the circle action given by restricting the $SU(2)$ action to T , then

$$\begin{aligned}\text{ind}_{SU(2)}^{\mu_n}(D) \\ = \frac{1}{2} (\text{ind}_T^{\rho_n}(D) + \text{ind}_T^{\rho_{-n}}(D) - \text{ind}_T^{\rho_{n+2}}(D) - \text{ind}_T^{\rho_{-n-2}}(D)),\end{aligned}\tag{5.1}$$

where the representation ρ_k satisfies

$$\rho_k(E(t)) = \text{multiplication by } e^{ikt}.$$

6. EXAMPLES

6.1. Signature and de Rham operators, torus action. Let M be a Riemannian manifold of dimension $2n$ endowed with an isometric action of T^m . Let \bar{x} be an isolated fixed point of this action. There exist geodesic normal coordinates $(z_1, \dots, z_n) \in \mathbb{C}^n$ and coordinates $\theta \in \mathbb{R}^m / 2\pi\mathbb{Z}^m$ for T^m such that $\bar{x} = (0, \dots, 0)$ and the action of θ is expressed using the vectors $\mathbf{k}_1 = (k_{11}, \dots, k_{1m}), \dots, \mathbf{k}_n = (k_{n1}, \dots, k_{nm}) \in \mathbb{Z}^m$ as follows:

$$(z_1, \dots, z_n) \mapsto (e^{i\mathbf{k}_1 \cdot \theta} z_1, \dots, e^{i\mathbf{k}_n \cdot \theta} z_n).$$

Let \mathbf{v} be an element of the Lie algebra of T^m such that

$$\theta(t) = \exp(t\mathbf{v}) = (t\tau_1, t\tau_2, \dots, t\tau_m) = t\tau \in T^m$$

generates a dense flow in T^m , so that the set $\{\tau_1, \dots, \tau_m\}$ must be linearly independent over \mathbb{Q} , and such that each $\tau_p > 0$, as in Section 3. Let V be the vector field on M generated by this action. Consider the operator $d + d^*$ on forms $\Gamma(M, \Lambda^* T^* M)$, and let $c : T^* M \rightarrow \text{End}(\Lambda^* T^* M)$ denote the standard Clifford action by cotangent vectors, so that the Dirac operator is

$$D = d + d^* = c \circ \nabla,$$

where ∇ is the Levi-Civita connection on forms.

From (3.5), we have

$$\kappa_q = \mathbf{k}_q \cdot \tau > 0, \quad 1 \leq q \leq n;$$

otherwise the orientation of the q^{th} plane needs to be reversed. Assume that this has been done.

Next we compute the numbers ε_{jq} from formula (3.7). We consider the Hermitian operators

$$ic(d\text{vol}_q) = ic(dx_q)c(dy_q)$$

evaluated at the fixed point. The vector space $E_{\overline{x}} = \Lambda^* T_{\overline{x}}^* M$ consists of forms

$$(A_1 + B_1 dx_1 + C_1 dy_1 + D_1 dx_1 \wedge dy_1) \wedge \dots \wedge (A_n + B_n dx_n + C_n dy_n + D_n dx_n \wedge dy_n),$$

where $A_q, B_q, C_q, D_q \in \mathbb{C}$ for each q . Observe that $ic(dx_q)c(dy_q)$ acts only on the q^{th} component of the wedge product above, and

$$\begin{aligned} & ic(dx_q)c(dy_q)(A_q + B_q dx_q + C_q dy_q + D_q dx_q \wedge dy_q) \\ &= i(A_q dx_q \wedge dy_q + B_q dy_q - C_q dx_q - D_q) \end{aligned}$$

Hence, the q^{th} components of eigenspaces of $ic(dx_q)c(dy_q)$ are

$$\begin{aligned} E_{\pm 1} &= \text{span}\{1 \pm i dx_q \wedge dy_q, dx_q \pm i dy_q\} \\ &= \begin{cases} \text{span}\{1 + \frac{1}{2} d\overline{z}_q \wedge dz_q, dz_q\} & \text{for eigenvalue } +1 \\ \text{span}\{1 - \frac{1}{2} d\overline{z}_q \wedge dz_q, d\overline{z}_q\} & \text{for eigenvalue } -1 \end{cases} \end{aligned}$$

Henceforth we choose a basis of the 4^n -dimensional space $E_{\overline{x}}$ as follows. Let

$$\omega_{1q} = 1 + \frac{1}{2} d\overline{z}_q \wedge dz_q; \quad \omega_{2q} = 1 - \frac{1}{2} d\overline{z}_q \wedge dz_q; \quad \omega_{3q} = dz_q; \quad \omega_{4q} = d\overline{z}_q$$

Then for each $\mathbf{i} = (i_1, \dots, i_n) \in \{1, 2, 3, 4\}^n$, we have the basis element $\omega_{\mathbf{i}}$ defined by

$$\omega_{\mathbf{i}} = \omega_{i_1 1} \wedge \dots \wedge \omega_{i_n n},$$

and $\{\omega_{\mathbf{i}} \mid \mathbf{i} \in \{1, 2, 3, 4\}^n\}$ forms a basis of $E_{\overline{x}}$. Note that an element $\theta \in T^m$ acts on

$$\left(\tilde{A}_1 \omega_{11} + \tilde{B}_1 \omega_{21} + \tilde{C}_1 \omega_{31} + \tilde{D}_1 \omega_{41} \right) \wedge \dots \wedge \left(\tilde{A}_n \omega_{1n} + \tilde{B}_n \omega_{2n} + \tilde{C}_n \omega_{3n} + \tilde{D}_n \omega_{4n} \right)$$

via

$$\tilde{A}_q \omega_{1q} + \tilde{B}_q \omega_{2q} + \tilde{C}_q \omega_{3q} + \tilde{D}_q \omega_{4q} \mapsto \tilde{A}_q \omega_{1q} + \tilde{B}_q \omega_{2q} + e^{-i\mathbf{k}_q \cdot \theta} \tilde{C}_q \omega_{3q} + e^{i\mathbf{k}_q \cdot \theta} \tilde{D}_q \omega_{4q}$$

The irreducible representation spaces are 1-dimensional spaces spanned by $\omega_{\mathbf{i}}$. The representation restricted to the span $\{\omega_{\mathbf{i}}\}$ is $\rho(\theta) = \text{multiplication by}$

$$\prod_{q, i_q=3} e^{-i\mathbf{k}_q \cdot \theta} \prod_{N, i_N=4} e^{+i\mathbf{k}_N \cdot \theta},$$

corresponding to the vector $\mathbf{a}_i \in \mathbb{Z}^m$ from (3.6) with

$$\mathbf{a}_i = - \sum_{i_q=3} \mathbf{k}_q + \sum_{i_q=4} \mathbf{k}_q$$

Next, we calculate the integers ε_{iq} from (3.7):

$$\varepsilon_{iq} \omega_i = ic(d\text{vol}_q) \omega_i = (-1)^{i_q+1} \omega_i, \text{ so } \varepsilon_{iq} = (-1)^{i_q+1}. \quad (6.1)$$

Theorem 4.2 implies that the contribution to the equivariant index of D at \bar{x} is an alternating sum of the quantities

$$k_i(\bar{x}) = \begin{cases} \# \{ \mathbf{m} \in \mathbb{Z}^n \mid m_h \leq 0 \text{ and } \sum_{h=1}^n m_h \mathbf{k}_h = \mathbf{a}_i + \mathbf{b} \} & \text{if } \varepsilon_{ih} = -1 \text{ for all } h \\ 0 & \text{otherwise} \end{cases}$$

We only count those $k_i(\bar{x})$ with i_h even for all $h \in \{1, \dots, n\}$. For these, we have

$$\begin{aligned} k_i(\bar{x}) &= \# \left\{ \mathbf{m} \in \mathbb{Z}^n \mid m_h \leq 0 \text{ and } \mathbf{b} = \sum_{h=1}^n \begin{cases} m_h \mathbf{k}_h & \text{if } i_h = 2 \\ (m_h - 1) \mathbf{k}_h & \text{if } i_h = 4 \end{cases} \right\} \\ &= \# \left\{ \mathbf{m} \in \mathbb{Z}^n \mid m_h \leq 0 \text{ and } \mathbf{b} \cdot \tau = \sum_{h=1}^n \begin{cases} m_h \kappa_h & \text{if } i_h = 2 \\ (m_h - 1) \kappa_h & \text{if } i_h = 4 \end{cases} \right\} \end{aligned} \quad (6.2)$$

The integer $\text{sign}(\mathbf{i})$ is ± 1 according to whether $\omega_i \in E_{\bar{x}}^+$ or $E_{\bar{x}}^-$. For example, if $\mathbf{b} = \mathbf{0}$ (that is, we restrict to invariant sections), then

$$k_i(\bar{x}) = \begin{cases} 1 & \text{if } i_h = 2 \text{ for every } h \in \{1, \dots, n\}. \\ 0 & \text{otherwise} \end{cases}$$

Therefore, if ρ_0 is the trivial representation,

$$\text{ind}_{T^m}^{\rho_0}(D) = \sum_{V(\bar{x})=0} \pm 1,$$

where the sign is determined by $\omega_{21} \wedge \dots \wedge \omega_{2n} \in E_{\bar{x}}^+$ or $E_{\bar{x}}^-$. Since $\omega_{21} \wedge \dots \wedge \omega_{2n}$ is an even form, for the de Rham operator we have

$$\text{Euler}(M)^{\rho_0} = \text{ind}_{T^m}^{\rho_0}(D) = \text{number of singular points of } V.$$

If we consider the signature operator, the chirality of $\omega_{21} \wedge \dots \wedge \omega_{2n}$ is $(-1)^n \text{sign}(V, \bar{x})$, where $\text{sign}(V, \bar{x})$ is ± 1 according to whether the orientation of M agrees with the orientation of our chosen coordinates (z_1, \dots, z_n) — that is, whether the orientation on the tangent space $T_{\bar{x}}M$ agrees with that induced from V . We write

$$\text{Signature}(M)^{\rho_0} = \text{ind}_{T^m}^{\rho_0}(D) = (-1)^n \sum_{V(\bar{x})=0} \text{sign}(V, \bar{x}).$$

Note, that the kernel of D consists of harmonic forms, which are always invariant under isometric actions of connected Lie groups, so that in fact the formulas above yield results

about the Euler characteristic $\chi(M)$ and signature:

$$\chi(M) = \text{number of singular points of } V \quad (6.3)$$

$$\text{Signature}(M) = (-1)^n \sum_{V(\bar{x})=0} \text{sign}(V, \bar{x}). \quad (6.4)$$

If our representation is not trivial, then $b_p \neq 0$ for some p , and

$$\mathbf{b} \cdot \tau = - \sum_{h=1}^n c_h \kappa_h$$

with integers $c_h \geq 0$ for all h (see (6.2)). If $c_h > 0$, both $i_h = 2$ and $i_h = 4$ yield positive values of $k_{\mathbf{i}}(\bar{x})$. Let A be a subset of $\{1, \dots, n\}$, and let

$$N(A, \mathbf{b}, \bar{x}) = \# \text{ of ways to write } \mathbf{b} \cdot \tau = - \sum_{h \in A} c_h \kappa_h$$

with $c_h \in \mathbb{Z}_{>0}$ for all $h \in A$. (6.5)

Further, let

$$S(A, \bar{x}) = \sum_{\mathbf{i} \in \mathbf{I}_A} \text{sign}(\mathbf{i}),$$

where the sum is taken over the set \mathbf{I}_A of all multi-indices $\mathbf{i} = (i_1, \dots, i_n)$ such that

$$i_h = \begin{cases} 2 \text{ or } 4 & \text{if } h \in A. \\ 2 & \text{otherwise} \end{cases},$$

and where

$$\text{sign}(\mathbf{i}) = \begin{cases} 1 & \text{if } \omega_{\mathbf{i}} \in E_{\bar{x}}^+ \\ -1 & \text{if } \omega_{\mathbf{i}} \in E_{\bar{x}}^- \end{cases},$$

Then the contribution of the critical point \bar{x} is

$$\sum_{\mathbf{i}} \text{sign}(\mathbf{i}) k_{\mathbf{i}}(\bar{x}) = \sum_{A \subset \{1, \dots, n\}} N(A, \mathbf{b}, \bar{x}) S(A, \bar{x}),$$

and thus

$$\text{ind}_{T^m}^{\rho_{\mathbf{b}}} (D) = \sum_{V(\bar{x})=0} \sum_{A \subset \{1, \dots, n\}} N(A, \mathbf{b}, \bar{x}) S(A, \bar{x}).$$

Note that if D is the de Rham operator and if $b_p \neq 0$ for some $p \in \{1, \dots, m\}$, then $S(A, \bar{x}) = 0$ for every singular point \bar{x} and every subset $A \subset \{1, \dots, n\}$. The reason is that replacing ω_{2q} with ω_{4q} or vice versa changes the parity of the form. Thus,

$$\text{Euler}(M)^{\rho_{\mathbf{b}}} = \text{ind}_{T^m}^{\rho_{\mathbf{b}}} (D) = 0, \quad (6.6)$$

which agrees with the fact that the kernel of D consists of invariant forms.

If D is the signature operator, then $\text{sign}(\mathbf{i})$ is the same for each of the indices such that $i_h = 2$ or 4 and is $\text{sign}(V, \bar{x})$. Thus, if $b_p \neq 0$ for some $p \in \{1, \dots, m\}$,

$$\begin{aligned} \text{Signature}(M)^{\rho_{\mathbf{b}}} &= \text{ind}_{T^m}^{\rho_{\mathbf{b}}} (D) \\ &= \sum_{V(\bar{x})=0} \text{sign}(V, \bar{x}) \sum_{A \subset \{1, \dots, n\}} 2^{|A|} N(A, \mathbf{b}, \bar{x}), \end{aligned} \quad (6.7)$$

which is (surprisingly) zero because the harmonic forms are invariant.

Proposition 6.1. *Given a Killing field V on an $2n$ -dimensional manifold with isolated singularities,*

$$\begin{aligned}\chi(M) &= \text{number of singular points of } V \\ \text{Signature}(M) &= (-1)^n \sum_{V(\bar{x})=0} \text{sign}(V, \bar{x})\end{aligned}$$

For any $\mathbf{b} \in \mathbb{Z}^m$, where V generates a T^m action,

$$\sum_{V(\bar{x})=0} \text{sign}(V, \bar{x}) \sum_{A \subset \{1, \dots, n\}} 2^{|A|} N(A, \mathbf{b}, \bar{x}) = 0.$$

Remark 6.2. *The first two identities above are well known. The first identity is a special case of the Hopf index theorem where the Hopf index of the vector field is one at each singular point. The factor $(-1)^n$ may be removed from the second identity, since the signature is zero if n is odd. Both of the first two identities are particular cases of the Atiyah-Bott fixed point formula. The third identity seems to be new.*

Example 6.3. *A particular example of the calculations above is the following action of T^{n+1} on complex projective space $\mathbb{C}P^n$. Consider homogeneous coordinates $[z_1, \dots, z_{n+1}]$ with the standard metric, and consider the family of isometries $[z_1, \dots, z_{n+1}] \mapsto [e^{i\theta_1} z_1, \dots, e^{i\theta_{n+1}} z_{n+1}]$, where $\theta = (\theta_1, \dots, \theta_{n+1}) \in T^{n+1}$. Let*

$$\theta(t) = \exp(t\mathbf{v}) = (t\tau_1, t\tau_2, \dots, t\tau_{n+1}) \in T^{n+1},$$

generates a dense flow in T^{n+1} so that the set $\{\tau_1, \dots, \tau_{n+1}\}$ is linearly independent over \mathbb{Z} and $0 < \tau_1 < \dots < \tau_{n+1}$. Let V be the vector field generated by this action.

There are $n+1$ fixed points: $[e_1] = [1, 0, \dots, 0]$, $[e_2] = [0, 1, 0, \dots, 0]$, \dots , and $[e_{n+1}] = [0, \dots, 0, 1]$. The homogeneous coordinates (in a coordinate chart diffeomorphic to \mathbb{C}^n) near $[e_l]$ are $[z_1, \dots, z_{l-1}, 1, z_{l+1}, \dots, z_{n+1}]$, and the action in these coordinates is

$$[z_1, \dots, z_{l-1}, 1, z_{l+1}, \dots, z_{n+1}] \mapsto [e^{i(\theta_1 - \theta_l)} z_1, \dots, 1, \dots, e^{i(\theta_{n+1} - \theta_l)} z_{n+1}].$$

The numbers k_{hp} in Formula 3.4 are

$$k_{hp} = \delta_{hp} - \delta_{pl},$$

where δ_{hp} is the Kronecker delta. From (3.5),

$$\kappa_q = \mathbf{k}_q \cdot \tau = \tau_q - \tau_l > 0, \quad 1 \leq q \leq n+1 \text{ and } q \neq l;$$

Thus the orientation of the q^{th} plane needs to be reversed if $q < l$. Thus we have

$$\text{sign}(V, [e_l]) = (-1)^{l+1}.$$

From Equations (6.3), (6.4), and (6.6), we have

$$\begin{aligned}\chi(\mathbb{C}P^n) &= n+1 \\ \text{Signature}(\mathbb{C}P^n) &= \sum_{l=1}^{n+1} (-1)^{l+1} = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \\ \chi(\mathbb{C}P^n)^{\rho_b} &= 0\end{aligned}$$

Formula (6.5) gives

$$\begin{aligned} \mathbf{b} \cdot \tau &= \sum_{h=1}^{l-1} c_h (\tau_h - \tau_l) - \sum_{h=l+1}^{n+1} c_h (\tau_h - \tau_l), \text{ so} \\ c_1 &= b_1; \dots; c_{l-1} = b_{l-1}; c_{l+1} = -b_{l+1}; \dots; c_{n+1} = -b_{n+1} \\ b_l &= \sum_{h=l+1}^{n+1} c_h - \sum_{h=1}^{l-1} c_h. \end{aligned}$$

Thus, the only irreducible representations $\rho_{\mathbf{b}}$ that could have nonzero contributions from the singular point $[e_l]$ are those which satisfy

$$\begin{aligned} b_1, \dots, b_{l-1} &\geq 0; b_{l+1}, \dots, b_{n+1} \leq 0; \text{ and} \\ b_l &= -\sum_{h \neq l} b_h. \end{aligned}$$

Since the integers c_h determine \mathbf{b} , we have

$$N(A, \mathbf{b}, [e_l]) = 1.$$

We give a specific example of a representation and the resulting formula; we leave it to the reader to obtain a general formula that works for all possible \mathbf{b} . Suppose

$$(b_1, \dots, b_{13}) = (0, 1, 0, 0, 76, 0, 0, 0, 0, 0, -51, -24, -2)$$

for the action of T^{13} on $\mathbb{C}P^{12}$. Then, by the computations above and formula (6.7), we have

$$\begin{aligned} \text{Signature}(M)^{\rho_{\mathbf{b}}} &= \sum_{V(\bar{x})=0} \text{sign}(V, \bar{x}) \sum_{A \subset \{1, \dots, n\}} 2^{|A|} N(A, \mathbf{b}, \bar{x}) \\ &= (-1)^6 2^4 + (-1)^7 2^5 + (-1)^8 2^5 + (-1)^9 2^5 + \\ &\quad + (-1)^{10} 2^5 + (-1)^{11} 2^5 + (-1)^{12} 2^4 \\ &= 0. \end{aligned}$$

6.2. An example of an $SU(2)$ -action. As in Example 5.1, let $T < SU(2)$ be the maximal torus defined as

$$T = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Consider the manifold $M = SU(2)/T$. We identify each $(\alpha, \beta) \in S^3 \subset \mathbb{C}$ with the corresponding matrix $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in SU(2)$. Each element of M is an equivalence class depending on $(\alpha, \beta) \in S^3 \subset \mathbb{C}$:

$$\begin{aligned} [(\alpha, \beta)] &= \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} = \begin{pmatrix} \alpha e^{it} & -\bar{\beta} e^{-it} \\ \beta e^{it} & \bar{\alpha} e^{-it} \end{pmatrix} \mid t \in \mathbb{R} \right\} \\ &= [(e^{it}\alpha, e^{it}\beta)]. \end{aligned}$$

We endow M with the standard metric and the left $SU(2)$ -action, which is

$$\begin{aligned} (z, w) [(\alpha, \beta)] &= \left[\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \right] \\ &= [(z\alpha - \bar{w}\beta, w\alpha + \bar{z}\beta)]. \end{aligned}$$

Then $M = S^2$ (because $(\alpha, \beta) \rightarrow [(\alpha, \beta)]$ is the Hopf fibration), so for example its Euler characteristic is 2 and its signature is zero. Consider the de Rham operator $D = d + d^*$ on forms; this operator commutes with the $SU(2)$ -action and with the even-odd grading. The kernel of D is the set of harmonic forms, which consist of constants and constants times the volume form. Both of these are invariant forms, so we have that (in the notation of Example 5.1)

$$\text{ind}_{SU(2)}^{\mu_n}(D) = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Since the de Rham operator is transversally elliptic with respect to the torus T action, we also have

$$\text{ind}_{SU(2)}^{\mu_n}(D) = \frac{1}{2} (\text{ind}_T^{\rho_n}(D) + \text{ind}_T^{\rho_{-n}}(D) - \text{ind}_T^{\rho_{n+2}}(D) - \text{ind}_T^{\rho_{-n-2}}(D))$$

by Equation 5.1. The T -action on M is given by

$$(e^{is}, 0) [(\alpha, \beta)] = [(e^{is}\alpha, e^{-is}\beta)] = [(e^{2is}\alpha, \beta)].$$

It has two fixed points, $[(1, 0)]$ and $[(0, 1)]$.

On one hand we know that the kernel of D consists of T -invariant forms, so that

$$\text{ind}_T^{\rho_n}(D) = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{Z}$. Thus, we have that if $n \geq 0$,

$$\begin{aligned} \frac{1}{2} (\text{ind}_T^{\rho_n}(D) + \text{ind}_T^{\rho_{-n}}(D) - \text{ind}_T^{\rho_{n+2}}(D) - \text{ind}_T^{\rho_{-n-2}}(D)) &= \begin{cases} \frac{1}{2} (1 + 1 - 0 - 0) & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \text{ind}_{SU(2)}^{\mu_n}(D), \end{aligned}$$

as expected.

To compute the index of D using Theorem 4.2, we let the vector field V be the infinitesimal generator of the action $[(\alpha, \beta)] \mapsto [(e^{is}\alpha, e^{-is}\beta)]$. Using the calculation in the previous section, the index $\text{ind}_T^{\rho_0}(D)$ is the number of singular points, which is two, and all other indices are zero, as expected.

Note that if we compute $\text{ind}_{SU(2)}^{\mu_n}$ or $\text{ind}_T^{\rho_n}$ of the spin Dirac operator, we obtain zero for all indices, by the Atiyah-Hirzebruch vanishing theorem.

Formula 5.1 does not apply when the operator is transversally elliptic with respect to the $SU(2)$ action but not to the T -action. For example, the zero operator

$$\mathbf{0} : \Gamma(SU(2) \backslash T, \mathbb{C}) \rightarrow \Gamma(SU(2) \backslash T, \{0\})$$

is equivariant and transversally elliptic with respect to the $SU(2)$ action and is equivariant but not transversally elliptic with respect to the T action. One may check that the μ_n part of $\ker \mathbf{0}$ is zero if n is odd otherwise is the eigenspace of the Laplacian on $S^2 = SU(2) \backslash T$ with eigenvalue $\frac{n}{2}$, which occurs with multiplicity 1. Thus,

$$\text{ind}_{SU(2)}^{\mu_n}(\mathbf{0}) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}.$$

for all $n \geq 0$. Note that in every irreducible representation space μ_n , the representations $\rho_n, \rho_{n-2}, \dots, \rho_{-n}$ of T occur, each with multiplicity 1. Thus the ρ_n part of $\ker \mathbf{0}$ is

$$\text{ind}_{SU(2)}^{\rho_n}(\mathbf{0}) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \infty & \text{if } n \text{ is even} \end{cases}.$$

This demonstrates that Equation 5.1 is valid only if the corresponding indices $\text{ind}_T^{\rho_n}(D)$ are finite, which happens always when D is T -transversally elliptic. Note that the formula remains valid even if D is not transversally elliptic if the corresponding ρ_n parts of the subspaces are finite dimensional, as above in the case where n is odd.

6.3. A transversally elliptic operator on the sphere.

6.3.1. *The operator D and its equivariant index.* Let $\alpha \in S^1$ act on

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

by a rotation of 2α around the z -axis. Let E be the trivial \mathbb{C}^2 bundle over S^2 . Let $\alpha \in S^1$ act on $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in E$ by $F_\alpha \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} w_1 \\ e^{i\alpha} w_2 \end{pmatrix}$. We use the grading $E^+ = \left\{ \begin{pmatrix} f \\ 0 \end{pmatrix} \right\}$, $E^- = \left\{ \begin{pmatrix} 0 \\ g \end{pmatrix} \right\}$. Consider the transversally elliptic operator D on sections of E over S^2 . We will write it in two ways, using rectangular coordinates (x, y, z) or spherical coordinates (θ, ϕ) with ϕ the angle between the position vector and the z -axis and θ the polar angle in the xy -plane. Let $\text{Proj}: T\mathbb{R}^3 \rightarrow TS^2$ be the orthogonal projection.

$$\begin{aligned} D &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{Proj} \left(-z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z} \right) + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \text{Proj} \left(-z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} \right) \\ &= \begin{pmatrix} 0 & e^{-i\theta} \\ -e^{i\theta} & 0 \end{pmatrix} \frac{\partial}{\partial \phi} + \cot \phi \begin{pmatrix} 0 & -ie^{-i\theta} \\ -ie^{i\theta} & 0 \end{pmatrix} \frac{\partial}{\partial \theta}. \end{aligned}$$

Note that this operator fails to be elliptic precisely at the equator $z = 0$ (or $\phi = \frac{\pi}{2}$). It is an easy exercise to check that this operator is S^1 -equivariant and symmetric for the standard metric on S^2 . It can be shown that these properties imply that D is essentially self-adjoint.

If $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \ker D$, then in the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$,

$$\begin{aligned} D \begin{pmatrix} u_1 \\ 0 \end{pmatrix} (x, y) &= \begin{pmatrix} 0 \\ (-z \frac{\partial u_1}{\partial x} + x \frac{\partial u_1}{\partial z}) + i \left(-z \frac{\partial u_1}{\partial y} + y \frac{\partial u_1}{\partial z} \right) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \left(-z \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \right) u_1(x, y) \end{pmatrix} = \mathbf{0}. \end{aligned}$$

Thus, u_1 must be holomorphic as a function of the coordinates (x, y) . Similarly, u_2 must be antiholomorphic. The same facts are true for the restriction of u to the lower hemisphere. Note that any (anti-)holomorphic function that is defined on the unit disk and continuous on the closure is determined by its values on the boundary. Thus, the function u_1 is symmetric with respect to the xy -plane, as is u_2 . We conclude that the smooth sections $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ in $\ker D$ are exactly functions of x and y alone such that u_1 is holomorphic and u_2 is antiholomorphic.

We now find the kernel of D restricted to the representation classes of the S^1 action. For $k \in \mathbb{Z}_{\geq 0}$, $w = x + iy$, and $\alpha \in S^1$,

$$\psi_\alpha \begin{pmatrix} w^k \\ 0 \end{pmatrix} = \psi_\alpha \begin{pmatrix} (\sin \phi)^k e^{ik\theta} \\ 0 \end{pmatrix} = \begin{pmatrix} (\sin \phi)^k e^{i\alpha} e^{ik(\theta+2\alpha)} \\ 0 \end{pmatrix} = e^{i(2k+1)\alpha} \begin{pmatrix} w^k \\ 0 \end{pmatrix};$$

similarly, $\psi_\alpha \begin{pmatrix} 0 \\ \overline{w}^k \end{pmatrix} = e^{-i(2k+1)\alpha} \begin{pmatrix} 0 \\ \overline{w}^k \end{pmatrix}.$

Thus, $\ker D$ is the direct sum of the irreducible representations of S^1 on $\ker D$ corresponding to $\alpha \mapsto$ multiplication by $e^{i(2k+1)\alpha}$ for $k \in \mathbb{Z}$. Then

$$\text{ind}^{\rho_n}(D) = \begin{cases} -1 & \text{if } n < 0 \text{ and } n \text{ is odd} \\ 1 & \text{if } n > 0 \text{ and } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases},$$

where ρ_n is the representation $\alpha \mapsto$ multiplication by $e^{in\alpha}$.

6.3.2. *The equivariant perturbation Z .* Next, we will verify Theorem 4.2 by calculating this same index using an equivariant perturbation Z . Let

$$Z = \sin \phi \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}.$$

We chose this Z so that at the north and south poles, it will agree with Clifford multiplication by $\pm i\partial_\theta$, as we shall soon see. The bundle map Z is equivariant; one may check that for any section u , $Z(\psi_\alpha u)(\theta, \phi) = \psi_\alpha(Zu)(\theta, \phi)$. Further, Z is nonsingular away from the north pole $\phi = 0$ and south pole $\phi = \pi$. Next, we see that $DZ + ZD$ is bounded on sections of the form

$$u = \begin{pmatrix} f(\phi) e^{im_1\theta} \\ g(\phi) e^{im_2\theta} \end{pmatrix},$$

because the coefficient of $\frac{\partial}{\partial \phi}$ in the expression $DZ + ZD$ is zero. Thus, it is bounded on sections of type ρ_n , where ρ_n is an irreducible representation of S^1 .

Since

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(-z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z} \right) + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \left(-z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} \right),$$

at $z = 1$ the operator is

$$D_{NP} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \frac{\partial}{\partial y},$$

and on the whole sphere

$$\begin{aligned} Z &= i \sin \phi \begin{pmatrix} 0 & -ie^{-i\theta} \\ -ie^{i\theta} & 0 \end{pmatrix} \\ &= i \begin{pmatrix} 0 & -ix - y \\ -ix + y & 0 \end{pmatrix} = -iy \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + ix \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \end{aligned}$$

At the north pole we define Clifford multiplication as

$$c_{NP} \left(a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} \right) := a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

then

$$Z = i c_{NP} \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) = i c_{NP} (\partial_\theta).$$

Similarly, at the south pole,

$$D_{SP} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial y},$$

and we define Clifford multiplication as

$$c_{SP} \left(a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} \right) := a \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

so that

$$Z = i c_{SP} \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) = i c_{SP} (-\partial_\theta).$$

We now use Theorem 4.2 to calculate the index $\text{ind}^{\rho_n}(D)$. We consider sections of type ρ_n . At the north pole $z = 1$, we have

$$\begin{aligned} \tau_1 &= 1; \mathbf{b} = b_1 = n; k_{11} = \kappa_1 = 2 \\ a_{11} &= -1; a_{21} = 1; \varepsilon_{11} = 1; \varepsilon_{21} = -1 \end{aligned}$$

Then the contribution to the index at the north pole is

$$\begin{aligned} \text{sign}(1) k_1(NP) + \text{sign}(2) k_2(NP) &= 0 - k_2(NP) \\ &= -\#\{m \in \mathbb{Z} \mid m \leq 0, \varepsilon_{21} = -1, \text{ and } mk_{11} = a_{21} + b_1\} \\ &= -\#\{m \in \mathbb{Z} \mid m \leq 0, \varepsilon_{21} = -1, \text{ and } 2m = 1 + n\} \\ &= -1 \text{ if } n < 0 \text{ and } n \text{ is odd.} \end{aligned}$$

At the south pole, with the orientation reversed on both the surface and on the group S^1 ,

$$\begin{aligned} \tau_1 &= 1; \mathbf{b} = b_1 = -n; k_{11} = \kappa_1 = 2 \\ a_{11} &= 1; a_{21} = -1; \varepsilon_{11} = -1; \varepsilon_{21} = 1 \end{aligned}$$

Then the contribution to the index at the south pole is

$$\begin{aligned} \text{sign}(1) k_1(SP) + \text{sign}(2) k_2(SP) &= k_1(NP) - 0 \\ &= \#\{m \in \mathbb{Z} \mid m \leq 0, \varepsilon_{11} = -1, \text{ and } mk_{11} = a_{11} + b_1\} \\ &= \#\{m \in \mathbb{Z} \mid m \leq 0, \varepsilon_{11} = -1, \text{ and } 2m = 1 - n\} \\ &= 1 \text{ if } n > 0 \text{ and } n \text{ is odd.} \end{aligned}$$

Then, as expected, we have

$$\text{ind}^{\rho_n}(D) = \begin{cases} -1 & \text{if } n < 0 \text{ and } n \text{ is odd} \\ 1 & \text{if } n > 0 \text{ and } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}.$$

7. APPENDIX

7.1. Proof of Lemma 3.1.

Proof. We have

$$(D_s)^2 - D^2 = s(ic(V) \circ D + iD \circ c(V)) + s^2 |V|^2.$$

Write $V = \sum V_j e_j$ in terms of a local orthonormal frame e_1, e_2, \dots of the tangent bundle, corresponding to geodesic normal coordinate vector fields $e_j = \partial_j$ at the origin of the coordinate system. At the origin of the coordinate system, we have

$$\begin{aligned} ZD + DZ &= ic(V) \circ D + iD \circ c(V) \\ &= i \sum_{j,k} V_j c(e_j) c(e_k) \nabla_k + i \sum_{j,k} c(e_k) \nabla_k \circ V_j c(e_j). \end{aligned}$$

Since $\nabla_k e_j = 0$ at the origin,

$$\begin{aligned} ZD + DZ &= i \sum_{j,k} V_j c(e_j) c(e_k) \nabla_k + i \sum_{j,k} V_j c(e_k) c(e_j) \nabla_k + i \sum_{j,k} c(e_k) c(e_j) \partial_k V_j \\ &= -2i \sum_j V_j \nabla_j - i \sum_j \partial_j V_j + i \sum_{j \neq k} c(e_k) c(e_j) \partial_k V_j, \end{aligned}$$

since $c(e_j) c(e_k) + c(e_k) c(e_j) = -2\delta_{jk}$. Then

$$\begin{aligned} ZD + DZ &= -2i\nabla_V - i \left(\operatorname{div}(V) - \sum_j V_j \sum_{k \neq j} \langle \nabla_k e_j, e_k \rangle \right) + i \sum_{j \neq k} c(e_k) c(e_j) \partial_k V_j \\ &= -2i\nabla_V - i \operatorname{div}(V) + i \sum_{j \neq k} c(e_k) c(e_j) \partial_k V_j \text{ since } \nabla_k e_j = 0 \text{ at the origin} \\ &= -2i\nabla_V - i \operatorname{div}(V) + ic(d(V^*)), \end{aligned}$$

where by $c(d(V^*))$ we imply that we have used the inverse of the symbol map σ to convert the two-form $d(V^*)$ to a Clifford algebra element. For example, $\sigma(e_1 e_2) = c(e_1) c(e_2) 1 = (dx_1 \wedge -dx_1 \lrcorner) (dx_2 \wedge -dx_2 \lrcorner) 1 = dx_1 \wedge dx_2$, so we define $c(dx_1 \wedge dx_2) = c(e_1 e_2)$ at the origin. Now, since the last expression is coordinate-free, we conclude that

$$ZD + DZ = -2i\nabla_V - i \operatorname{div}(V) + ic(d(V^*))$$

at all points. □

7.2. Proof of the statement in Example 3.5.

Proof. A Killing vector field X can be lifted to a vector field \overline{X} on the frame bundle, so that \overline{X} covers X and is invariant under the $SO(n)$ bundle. The vector field \overline{X} lifts uniquely to a vector field \widehat{X} on the principal spin bundle \widetilde{F} . Thus it acts on any bundle associated to \widetilde{F} , such as the spin bundle. Let X be an infinitesimal isometry. If g is the metric tensor, then $\mathcal{L}_X g = 0$. If Y, Z are any two tensor fields of the same type, then

$$\begin{aligned} X \langle Y, Z \rangle &= \langle \mathcal{L}_X Y, Z \rangle + \langle Y, \mathcal{L}_X Z \rangle \\ &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle. \end{aligned}$$

Thus $A_X = \mathcal{L}_X - \nabla_X$ is skew-symmetric and of degree zero, since $\langle A_X Y, Z \rangle = -\langle Y, A_X Z \rangle$. Hence its action on $\Gamma(TM)$ comes from the endomorphism (also called A_X) of TM . Choose a basis $\{e_i\}$ of $T_x M$, and we may identify A_X with the element $a_X \in \mathfrak{o}(n)$ by identifying

$T_x M$ with \mathbb{R}^n using the basis . Under this identification the antisymmetric matrix $a_X = ((a_X)_{ij})$ corresponds to an endomorphism $A_X = \frac{1}{4} \sum (a_X)_{ij} e_i e_j$ (see Lemma 4.8 in [16] for calculations).

Let $\lambda : \text{Spin}(n) \rightarrow SO(n)$ be the double cover, and let $d\lambda : \mathfrak{spin}(n) \rightarrow \mathfrak{o}(n)$ be the differential map on the Lie algebras. Observe that $\mathfrak{spin}(n) \cong \text{Cl}_2(\mathbb{R}^n)$, and the Lie bracket induced on $\text{Cl}_2(\mathbb{R}^n)$ is $[a, b] = ab - ba$ (using Clifford multiplication). For all $v \in \mathbb{R}^n$, $z \in \text{Spin}(n)$,

$$d\lambda(z)(v) = zv - vz,$$

where z is thought of as an element of $\text{Cl}_2(\mathbb{R}^n)$ and v is thought of as an element of $\text{Cl}_1(\mathbb{R}^n)$. Hence

$$\begin{aligned} d\lambda\left(\frac{1}{4} \sum (a_X)_{ij} e_i e_j\right)(v) &= \left[\frac{1}{4} \sum (a_X)_{ij} e_i e_j, v\right] \\ &= a_X v, \end{aligned}$$

so that

$$A_X = \mathcal{L}_X - \nabla_X = \frac{1}{4} \sum (a_X)_{ij} e_i e_j.$$

Next, given a Killing field X and vector field Y ,

$$\begin{aligned} A_X Y &= \mathcal{L}_X Y - \nabla_X Y \\ &= [X, Y] - [X, Y] - \nabla_Y X \\ &= -\nabla_Y X. \end{aligned}$$

Thus, given any vector field Z ,

$$\langle A_X Y, Z \rangle = -\langle \nabla_Y X, Z \rangle,$$

so $A_X = -(\nabla X)^\#$. This implies

$$(a_X)_{ij} = -\langle \nabla_{e_i} X, e_j \rangle.$$

Thus,

$$\begin{aligned} A_X &= \mathcal{L}_X - \nabla_X = -\frac{1}{4} \sum \langle \nabla_{e_i} X, e_j \rangle e_i e_j \\ &= -\frac{1}{4} \sum e_i (\langle \nabla_{e_i} X, e_j \rangle e_j) \\ &= -\frac{1}{4} \sum e_i (\nabla_{e_i} X) \\ &= -\frac{1}{4} \sum e_i (\partial_i X_j) e_j \text{ if } \{e_j\} \text{ is isochronous} \\ &= -\frac{1}{4} c(d(X^*)) \end{aligned}$$

We have therefore that

$$\mathcal{L}_X = \nabla_X - \frac{1}{4} c(d(X^*))$$

if X is a Killing vector field. □

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